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Department of Mathematics

MATH 2055 Tutorial 6 (Oct 28)

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1. True or False.

   (a) Let $f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

   $\because \lim_{n\to\infty} |\frac{1}{n}| = 0 \neq 1$

   $\therefore \lim_{n\to 0} f(x)$ does not exist.

   Solution: False

   By definition of limit of function, we don't need to consider the value of f at 0

   $\forall x$ which $0 < |x - 0| < \epsilon$, $|f(x)| = |x| < \epsilon$

   and therefore $\lim_{n\to 0} f(x) = 0$

   (b) Let f be a uniformly continuous function

   $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall$ x, y which $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$

   Pick any x', $y' \in \mathbb{R}$, WLOG, assume $x' < y'$

   if $|y' - x'| = (n + r)\delta$ where $r \in [0, 1)$

$$\begin{aligned}
|f(x') - f(y')| &= |f(x') - f(x' + \frac{y'-x'}{n+1}) + f(x' + \frac{y'-x'}{n+1}) - f(y')| \\
&\le |f(x') - f(x' + \frac{y'-x'}{n+1})| + |f(x' + \frac{y'-x'}{n+1}) - f(y')| \\
&< \epsilon + |f(x' + \frac{y'-x'}{n+1}) - f(y')| \\
&= \epsilon + |f(x' + \frac{y'-x'}{n+1}) - f(x' + \frac{2(y'-x')}{n+1}) + f(x' + \frac{2(y'-x')}{n+1}) - f(y')| \\
&< 2\epsilon + |f(x' + \frac{2(y'-x')}{n+1}) - f(y')| \\
&\vdots \\
&< (n+1)\epsilon = (\frac{(n+1)\epsilon}{(n+r)\delta})((n+r)\delta) < (\frac{2\epsilon}{\delta})((n+r)\delta) \\
&\le (\frac{2\epsilon}{\delta})|y' - x'|
\end{aligned}$$

$\therefore \exists$ constant M such that $\forall x''$, $y'' \in \mathbb{R}$, $|f(x'') - f(y'')| < M|x'' - y''|$

Solution: False

There are trouble when $|x'' - y''| < \delta$, ie, $n = 0$

The second last inequality is wrong.
This question show that uniformly continuity cannot give a bound on the " slope "

counter example: $f(x) = \sqrt(x)$ on $[0, \infty)$

if M exists, WLOG, we can assume $M > 1$,

$|f(\frac{1}{M^2}) - f(0)| = \frac{1}{M} > M|\frac{1}{M^2} - 0|$

which lead to contradiction.

But f is uniformly continuous.

As f is continuous on $[0, 1]$, therefore f is uniformly continuous on $[0, 1]$

$\forall \epsilon > 0$, $\exists \delta_1$ such that for all $x, y \in [0, 1]$ where $|x - y| < \delta_1$,

we have $|f(x) - f(y)| < \epsilon/2$

on $[0, \infty)$, for all $x, y \in [0, \infty)$ where $|x - y| < \epsilon/2$,

we have $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = |\frac{x - y}{\sqrt{x} + \sqrt{y}}| < |x - y| < \epsilon/2$

let $\delta = min\{\delta_1, \epsilon\}$

Pick any $x', y' \in [0, \infty)$ where $|x' - y'| < \delta$,

by above argument, if both $x', y' \in [0, 1]$ or both $x', y' \in [1, \infty)$, we have $|f(x') - f(y')| < \epsilon$

WLOG, we can assume $x' < y'$, if $x' \in [0, 1]$ and $y' \in [1, \infty)$

$$\begin{aligned}|f(x') - f(y')| &= |f(x') - f(1) + f(1) - f(y')| \\ &\leq |f(x') - f(1)| + |f(1) - f(y')| \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

therefore f is uniformly continuous on $[0, \infty)$

2. If f is a periodic continuous function ($\exists$ constant T such that $f(x) = f(T + x)$) , then f is uniformly continuous.

Solution:

As f is continuous on $[0, T]$, therefore f is uniformly continuous on $[0, T]$

$\forall \epsilon > 0$, $\exists \delta$ such that for all $x, y \in [0, T]$ where $|x - y| < \delta$,

we have $|f(x) - f(y)| < \epsilon/2$

WLOG, assume $\delta < T$

$\forall x'', y'' \in \mathbb{R}$ such that $|x'' - y''| < \delta$, there are only 2 cases

Case 1, there exists natural number n such that $x'', y'' \in [nT, (n + 1)T]$

$x'' - nT, y'' - nT \in [0, T]$

$|f(x'') - f(y'')| = |f(x'' - nT) - f(y'' - nT)| < \epsilon/2$

Case 2, WLOG, we can assume $x'' < y''$. if there exists natural number n' such that $x'' \in [(n'-1)T, n'T]$ and $y'' \in [n'T, (n'+1)T]$

$$\begin{aligned} |f(x'') - f(y'')| &= |f(x'') - f(n'T) + f(n'T) - f(y'')| \\ &\leq |f(x'') - f(n'T)| + |f(n'T) - f(y'')| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

therefore f is uniformly continuous.

3. Let $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ where m, n are relatively prime} \\ 0 & \text{if x is irrational} \end{cases}$

   Prove that f is continuous at 0.

   Solution:

   $\forall \epsilon > 0$, $\forall x$ where $|x - 0| < \epsilon$

   case 1, if x is irrational or 0,

   $|f(x) - f(0)| = 0 < \epsilon$

   case 2, if $x = \frac{m}{n}$ where m, n are relatively prime

   $|f(x) - f(0)| = |\frac{1}{n}| \leq |\frac{m}{n}| = |x - 0| < \epsilon$

   therefore f is continuous at 0

4. Let $\{f_k\}$ be a sequence of function and f is a function, such that

   $\forall x, \lim_{k \to \infty} f_k(x) = f(x)$

   Moreover, $\forall \epsilon > 0$, $\exists \delta$, such that $\forall k$, if $|x - y| < \delta$,

   then $|f_k(x) - f_k(y)| < \epsilon$

   Prove that f is uniformly continuous.

Solution:

the idea is that for fixed 2 point x and y, we can find large enough N , such that the function value are near at x and y

$\forall \epsilon > 0$, $\exists \delta$ such that $\forall k$, if $|x - y| < \delta$, then $|f_k(x) - f_k(y)| < \epsilon/3$

now, x and y are fixed.

because $\lim_{k\to\infty} f_k(x) = f(x)$

therefore $\exists N_1$ such that $\forall p \geq N_1$, $|f_p(x) - f(x)| < \epsilon/3$

because $\lim_{k\to\infty} f_k(y) = f(y)$

therefore $\exists N_2$ such that $\forall q \geq N_2$, $|f_q(y) - f(y)| < \epsilon/3$

let $N = max\{N_1, N_2\}$,

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}$$

therefore f is uniformly continuous